Multiscale Wavelet-LQR Controller for Linear Time Varying Systems

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Abstract: This paper proposes a multiresolution based wavelet controller for the control of linear time varying systems consisting of a time invariant component and a component with zero mean slowly time varying parameters. The real time discrete wavelet transform controller is based on a time interval from the initial until the current time and is updated at regular time steps. By casting a modified optimal control problem in a linear quadratic regulator (LQR) form constrained to a band of frequency in the wavelet domain, frequency band dependent control gain matrices are obtained. The weighting matrices are varied for different bands of frequencies depending on the emphasis to be placed on the response energy or the control effort in minimizing the cost functional, for the particular band of frequency leading to frequency dependent gains. The frequency dependent control gain matrices of the developed controller are applied to multiresolution analysis (MRA) based filtered time signals obtained until the current time. The use of MRA ensures perfect decomposition to obtain filtered time signals over the finite interval considered, with a fast numerical implementation for control application. The proposed controller developed using the Daubechies wavelet is shown to work effectively for the control of free and forced vibration (both under harmonic and random excitations) responses of linear time varying single-degree-of-freedom and multidegree-of-freedom systems. Even for the cases where the conventional LQR or addition of viscous damping fails to control the vibration response, the proposed controller effectively suppresses the instabilities in the linear time varying systems.

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Introduction

Several civil and mechanical engineering systems possess time varying system properties. These include disks mounted on vertical shafts with nonuniform elasticity, rotating machineries with cracks or nonuniform flexibility, cable stayed structures, offshore structures, variable speed wind turbines and helicopter blades to name a few. Such systems often exhibit instabilities including parametric and internal resonances and the associated dynamics have been dealt with in detail by Den Hartog (1956), Ibrahim (1985), and Dimentberg (1988).

The class of systems which shows variability in elasticity or stiffness variation due to opening and closing of cracks in civil engineering structures or cyclostationarity due to rotating machineries/turbo generators may lead to instabilities in the dynamic response (Den Hartog 1956). The nature of vibration of these systems inherently becomes nonstationary due to the introduction of additional frequencies (unlike a linear system) with the onset of the instabilities. While the dynamics of the stiffness/elasticity varying systems have been studied in literature, the control of such vibrations using active or semiactive control techniques will be the natural step to follow up based on the available understanding of these systems.

There has been limited amount of research available in literature on the control of time varying systems. Algebraic methods have been used by Kamen (1988) to control linear time varying systems. Tsakalis and Ioannou (1993) have presented results for the adaptive control of time varying systems. A robust adaptive control structure derived from the linear quadratic problem has been proposed by Sun and Ioannou (1992) and robust adaptive control has been dealt with in general by Ioannou and Sun (1996).

Since the vibratory signals of the previously mentioned civil and mechanical systems with variable stiffness are nonstationary in nature, a control law designed based on the time-frequency characteristics of the vibration signals is expected to control the instabilities better. To this end, a modified form of the conventional linear quadratic regulator (LQR) with the control gain derived by the use of wavelet analysis of the states is proposed in this paper. Wavelet analysis being a time-frequency technique is able to incorporate the information of the local time varying frequency content of the vibration signal. Hence, this can account for the instabilities which are known to be induced in certain frequency bands. Therefore, the weightings for the conventional LQR controller can be adjusted depending on the desired frequency bands required to be suppressed. Being a time-frequency technique, the wavelet based controller suppresses the frequencies locally in time. This wavelet-LQR controller works at different or multiple time scales, finally leading to a time varying control gain, even though for each frequency band width or scale the

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gains are time invariant. The control gain is formulated in the wavelet domain to manipulate the effects in the time-frequency domain. Finally, to compute the control in the time domain a multiresolution analysis (MRA) based discrete wavelet transform (DWT) is used, with the application of frequency band dependent gains to different filtered signals at different frequency bands. The use of MRA based DWT provides exact decomposition/reconstruction of signals and a fast algorithm for the purpose of control. The real time DWT controller is based on a window from the initial time, t_0, until the current time, t_c, with updating at regular time intervals. The formulation assumes that the parameters of system vary slowly in time.

Some examples of stiffness varying single-degree-of-freedom (SDOF) and two-degree-of-freedom (2DOF) systems have been considered. The systems have been subjected to free and forced vibrations with harmonic and random nonstationary excitations. The results show that the proposed controller is effective in suppressing the instabilities and controlling the vibrations. Comparison with the classical LQR shows that the proposed controller is even effective in cases where the former is unsuccessful in controlling the response.

Formulation

Let us consider a linear time varying system with a controller represented by state-space matrix equations as follows:

$$\{\dot{x}\} = [A(t)]\{x\} + [B]\{u\} + \{F\} \tag{1}$$

In Eq. (1), $\{x\}=(n\times 1)$ state vector; $[A(t)]=[A_0]+[\Delta A(t)]=(n\times n)$ time varying state matrix with $[A_0]$ and $[\Delta A(t)]$ as a time invariant (nominal) and a slowly time varying component, respectively; $[B]=(n\times m)$ control influence vector; $\{u\}=(m\times 1)$ control vector; and $\{F\}=(n\times 1)$ external excitation vector. On wavelet transforming and integrating by parts the ith equation in wavelet domain is (for standard results on wavelet analysis refer to Daubechies 1992)

$$\frac{\partial}{\partial b}W_{\psi}x_i(a,b) = \sum_{k=1}^{n} W_{\psi}(A_{ik}x_k)(a,b) + \sum_{k=1}^{m} B_{ik}W_{\psi}u_k(a,b) + W_{\psi}F_i(a,b); \quad \forall\ a \in R \tag{2}$$

where ψ=wavelet basis function and $W_{\psi}(\bullet)(a,b)$=wavelet transform of $(\bullet)$ with respect to the basis ψ. For a particular value of "a," Eq. (2) leads to an ordinary differential equation

$$W'_{\psi_a}x_i(b) = \sum_{k=1}^{n} W_{\psi_a}(A_{ik}x_k)(b) + \sum_{k=1}^{m} B_{ik}W_{\psi_a}u_k(b) + W_{\psi_a}F_i(b) \tag{3}$$

where prime denotes differentiation with respect to the parameter b (the translational parameter).

Consider the term $W_{\psi_a}(A_{ik}x_k)(b)$ in Eq. (3)

$$W_{\psi_a}(A_{ik}x_k)(b) = \frac{1}{\sqrt{a}}\int_{t_0}^{t_c} A_{ik}(t)x_k(t)\psi\left(\frac{t-b}{a}\right)dt; \quad t_0 \leqslant b \leqslant t_c \tag{4}$$

It may be noted that $\psi[(t-b)/a]$ is a fast decaying function localized around $(t=b)$, by the property of wavelet basis functions. If $A_{ik}(t)$ is a slowly varying function (with finite or countably infinite discontinuitites, e.g., sudden change in system parameters) as compared to $\psi[(t-b)/a]$ [i.e., if $|\hat{\psi}(a\omega)|$ (hat denotes a Fourier transformed quantity) is of higher frequency content] and/or $x_k(t)$, then for the evaluation of the integral in Eq. (4), $A_{ik}(t)$ can be approximated to be a constant with a value equal to the mean (nominal) value of A_{0ik}. This leads to

$$W_{\psi_a}(A_{ik}x_k)(b) \approx A_{0ik}W_{\psi_a}x_k(b) \tag{5}$$

Substituting Eq. (5), Eq. (3) for the ith state becomes

$$W'_{\psi_a}x_i(a,b) = \sum_{k=1}^{n} A_{0ik}W_{\psi_a}x_k(b) + \sum_{k=1}^{m} B_{ik}W_{\psi_a}u_k(b) + W_{\psi_a}F_i(b) \tag{6}$$

In matrix form, Eq. (6) can be expressed as

$$\{W'_{\psi_a}x\} = [A_0]\{W_{\psi_a}x\} + [B]\{W_{\psi_a}u\} + \{W_{\psi_a}F\}; \quad \forall\ a \in R \tag{7}$$

Eq. (7) is analogous to Eq. (1) in the wavelet domain with the time parameter being replaced by the translation parameter (around which temporal information is also localized) in the wavelet domain. The other difference between the two equations being that Eq. (7) is for a transformed process of the state i.e., $\{W_{\psi_a}x\}$ and not the state $\{x\}$ itself. From the modulus of the Fourier transform of $\{W_{\psi_a}x_i(b)\}$, i.e.,

$$|\hat{W}_{\psi_a}X_i(\omega)| = \sqrt{a}|\hat{\psi}(a\omega)||\hat{X}_i(\omega)| \tag{8}$$

it can be inferred that $W_{\psi_a}x_i(b)$ is narrow banded as $|\hat{\psi}(a\omega)|$ is narrow banded with localized frequency (by construction of wavelet basis), even though $x(t)$ may not be narrow banded. Hence, Eq. (1) has been transformed to a set of equations with states having narrow banded frequency content.

Wavelet Controller

The control action is expressed in wavelet domain as

$$\{W_{\psi_a}u\} = -[G]_a\{W_{\psi_a}x\} \tag{9}$$

where $[G]_a$=control gain matrix and is dependent on the dilation parameter a which controls the frequency content of $\{W_{\psi_a}u\}$. Hence, the gain matrix $[G]_a$ can be chosen depending upon the frequency bands over which the controller is desired to be acting (i.e., with a requirement of higher demand on the control force or effort to control the response) and can be varied for different frequency bands.

With the control equation given by Eq. (9), an alternative optimal control problem is formulated to sought the minimization of the functional

$$J_a = \int_{t_0}^{t_c} [\{W_{\psi_a}x\}^T[Q]_a\{W_{\psi_a}x\} + \{W_{\psi_a}u\}^T[R]_a\{W_{\psi_a}u\}]db \tag{10}$$

Eq. (10) is a quadratic functional as in case of a classical LQR but valid for wavelet transformed states at a frequency band with dilation parameter a. The matrices $[Q]_a$ and $[R]_a$=weighting matrices and are dependent on the parameter a corresponding to a frequency band. Hence, this makes it possible to vary the weighting matrices for different frequency bands if desired.

Interpretation of the Minimizing Functional

To interpret the physical significance of the functional in Eq. (10), let us consider a single term in the integrand of Eq. (10) arising out of the matrix multiplication $\{W_{\psi_a}x\}^T[Q]_a\{W_{\psi_a}x\}$, i.e., $W_{\psi_a}x_iQ_{ik_a}W_{\psi_a}x_k$. It can be shown that

$$C_\psi \int_0^\infty \frac{1}{a^2} \int_{t_0}^{t_c} W_{\psi_a} x_i Q_{ij_a} W_{\psi_a} x_k \, db\, da = T(x_i, x_k) \tag{11}$$

where Q_{ik_a}=element of $[Q]_a$ and $T(x_i,x_k)$ is a functional of x_i and x_k. If $[Q]_a$ and $[R]_a$ are assumed to be invariant with the frequency bands, i.e.

$$[Q]_a = [Q] \tag{12}$$

and

$$[R]_a = [R] \tag{13}$$

then, it follows that (Daubechies 1992):

$$T(x_i, x_k) = \int_{t_0}^{t_c} x_i(t) Q_{ik} x_k(t) dt \tag{14}$$

and

$$\int_0^\infty \frac{C_\psi}{a^2} J_a da = \int_{t_0}^{t_c} [\{x\}^T [Q] \{x\} + \{u\}^T [R] \{u\}] dt = J \tag{15}$$

which is the functional minimized for the classical LQR involving the combination of cost of the response and the control. Hence, the proposed optimal control problem formulation minimizing J_a in Eq. (10) minimizes the weighted combined cost of the response and the control effort in the frequency band corresponding to the parameter a. Though this is not the global optimal, it is a local optimal solution constrained to the frequency band concerned and thus is a constrained suboptimal problem.

Synthesis of the Control in Time Domain

To synthesize the control action in time domain at the time instant, $t=t_c$, based on the information available on the states in the time interval $[t_0,t_c]$, the use of continuous wavelet transform is not suitable. For exact decomposition/reconstruction of signals over a finite interval $[t_0,t_c]$ without any edge effects the use of DWT is essential. Hence, DWT will be used to synthesize the control, as derived in this section. In fact, the filtered signals (containing the information from wavelet coefficients) obtained from MRA based DWT will be used for the formulation in this section, instead of the wavelet coefficients directly. This also naturally eliminates the necessity of any integral calculations in evaluating the wavelet coefficients.

In order to compute the control action in real time, a relation between the continuous wavelet transform based control algorithm (as discussed in the previous section) and the DWT based control algorithm to be used for the proposed control scheme has to be established first. Hence, initially we consider the control in continuous time domain obtained from the continuous wavelet transform using the inversion theorem

$$\{u(t)\} = C_\psi^{-1} \int \int \frac{1}{a^2} \{W_{\psi_a} u(b)\} \psi\left(\frac{t-b}{a}\right) db\, da \tag{16}$$

Using Eq. (16) in Eq. (9)

$$\begin{aligned}\{u(t_c)\} = &- C_\psi^{-1} \int_0^{a_L} \int_{t_0}^{t_c} \frac{[G]_a}{a^2} \{W_{\psi_a} x(b)\} \psi\left(\frac{t_c-b}{a}\right) db\, da \\ &- C_\psi^{-1} \int_{a_L}^{a_u} \int_{t_0}^{t_c} \frac{[G]_a}{a^2} \{W_{\psi_a} x(b)\} \psi\left(\frac{t_c-b}{a}\right) db\, da \\ &- C_\psi^{-1} \int_{a_u}^{\infty} \int_{t_0}^{t_c} \frac{[G]_a}{a^2} \{W_{\psi_a} x(b)\} \psi\left(\frac{t_c-b}{a}\right) db\, da\end{aligned} \tag{17}$$

where a_L=dilation parameter below which the signal can be represented by a low frequency approximation and a_u=parameter which corresponds to the band above which are the frequency bands which could be ignored (i.e., these frequencies are not in the space of the function considered). Thus, the third term on the right hand side of Eq. (17) can be ignored. On sampling the dilation parameter a and discretizing we get a sequence $\{0, a_1, \ldots, a_L, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_u\}$. It is assumed that the different gain matrices vary in different bands according to

$$[G]_a = [G]_L; \quad a < a_L$$

$$[G]_a = [G]_j; \quad a_j < a < a_{j+1}$$

$$[G]_a = 0; \qquad a > a_u \tag{18}$$

where $[G]_a$=$[G]_j$ for all $a_L < a_j < a_{u-1}$=matrices which are constant over the band of frequencies for a particular scale a_j. Using Eq. (18) in Eq. (17)

$$\begin{aligned}\{u(t_c)\} = &- [G]_L C_\psi^{-1} \int_0^{a_L} \int_{t_0}^{t_c} \frac{1}{a^2} \{W_{\psi_a} x(b)\} \psi\left(\frac{t_c-b}{a}\right) db\, da \\ &- \sum_{j=L}^{u-1} [G]_j C_\psi^{-1} \int_{a_j}^{a_{j+1}} \int_{t_0}^{t_c} \frac{1}{a^2} \{W_{\psi_a} x(b)\} \psi\left(\frac{t_c-b}{a}\right) db\, da\end{aligned} \tag{19}$$

However, Eq. (19) will not be used for computation of control action. For the purpose of synthesis of the control function by using the filtered signals [to be used in Eq. (19)] for different frequency bands, DWT is used with perfect reconstruction capability. The DWT also lends itself to a fast numerical algorithm based on MRA. The idea behind the MRA using wavelets is very similar to subband decomposition where a signal is divided into a set of signals each containing a frequency band. In MRA the input at each stage is always split into two bands in time; the higher band becomes one of the outputs, while the lower band again is further split into two bands. This procedure is continued until a desired resolution is achieved.

Considering the state vector and using an appropriate wavelet with basis and scaling function ψ and ϕ, respectively, the scale equations are used to generate the high and low pass filters. There are two digital filters g and h used in the process of MRA, which determine the wavelet basis function $\psi(t)$ and the associated scaling function $\phi(t)$. For a dyadic wavelet construction, these two functions are given by two-scale equations

$$\phi(t) = 2\sum_{l} \bar{g}(l)\phi(2t-l) \tag{20}$$

and

$$\psi(t) = 2\sum_{l} \bar{h}(l)\phi(2t-l) \tag{21}$$

where, for perfect reconstruction, the coefficients $\bar{g}$ and $\bar{h}$, respectively, must satisfy the relationship

$$g(l) = \bar{g}(p-1-l) \tag{22}$$

$$h(l) = \bar{h}(p-1-l) \tag{23}$$

In Eqs. (22) and (23), the delay $p-1$=filter order for the chosen filter, which is related to the wavelet basis function.

These filters for a dyadic MRA with 2^n data points in the state signal are subsequently used to generate:

1. A low frequency signal approximation (below frequencies corresponding to the dilation a_L) which in terms of continuous wavelet transform can be written as

$$\{x\}_L = C_\psi^{-1}\int_0^{a_L}\int_{t_0}^{t_c}\frac{1}{a^2}\{W_{\psi_a}x(b)\}\psi\left(\frac{t-b}{a}\right)dbda \tag{24}$$

and is computed using the scaling function as

$$\{x\}_L = \int_{t_0}^{t_c}\{x(t)\}\phi_{Ln}(t)dt \tag{25}$$

2. Band limited signal components to give filtered signals in the frequency bands covered between the range corresponding to the dilation parameters a_j and a_{j+1}, as

$$\{x\}_{d_j} = C_\psi^{-1}\int_{a_j}^{a_{j+1}}\int_{t_0}^{tc}\frac{1}{a^2}\{W_{\psi_a}x(b)\}\psi\left(\frac{t-b}{a}\right)dbda \tag{26}$$

and is computed by

$$\{x\}_{d_j} = \int_{t_0}^{t_c}\{x(t)\}\psi_{jn}(t)dt \tag{27}$$

with

$$\phi_{Ln}(t) = 2^{-(L/2)}\phi\left(\frac{t}{2^L}-n\right) \tag{28}$$

$$\psi_{jn} = 2^{-[(j-2)/2]}\sum h(l)\phi\left(\frac{t}{2^{j-1}}-p+1-2^n\right) \tag{29}$$

The components from 1 and 2 can be used to reconstruct the signal

$$\{x(t)\} = \{x(t)\}_L + \sum_{j=L}^{u-1}\{x(t)\}_{d_j} \tag{30}$$

When Eqs. (24) and (26) are used in Eq. (19), it produces the control in time domain

$$\{u\} = -[G]_L\{x\}_L - \sum_{j=L}^{u-1}[G]_{d_j}\{x\}_{d_j} \tag{31}$$

synthesized using frequency dependent gains for different frequency bands. It may be noted that if the gain matrices in Eq. (31) are equal, i.e.

$$[G]_L = [G]_{d_j} = [G]; \quad \forall\, j = L, L+1, \ldots, u-1 \tag{32}$$

then it leads to the classical control.

The proposed MRA wavelet controller by the use of frequency dependent gains applied to the filtered time-frequency signals produces an equivalent gain with time varying nature. The control input can thus be alternatively expressed as

$$\{u\} = [G_e(t)]\{x\} \tag{33}$$

where the equivalent gain matrix is given by

$$[G_e(t)] = -[G]_L\{x\}_L\{x\}^T(\{x\}\{x\}^T)^{-1} - \sum_{j=L}^{u-1}[G]_{d_j}\{x\}_{d_j}\{x\}^T(\{x\}\{x\}^T)^{-1} \tag{34}$$

However, Eq. (34) is not used for calculation or implementation of the controller. Instead the control action is calculated in a more computationally simple way using Eq. (31). The frequency band dependent gains used in Eq. (31) are computed offline using the nominal state matrix $[A_0]$ and $[\Delta A(t)]$ is not required for the computation. The use of filtered time-frequency signals in real time to compute the control input in Eq. (31) and the multiplication of the time-frequency signals with the frequency dependent gains leading to an equivalent time varying control gain inherently accounts for the evolutionary frequency content of the response of the time varying system.

Determination of the Weighting Matrices

To obtain the gain matrices for different frequency bands, it is necessary to determine the weighting matrices for different bands to solve the optimal control problem for that frequency band. With the discretized bands, the weighting matrices will be $[Q]_L$, $[R]_L$ and $[Q]_j$, $[R]_j$; for $L<j<u-1$. The weighting matrices may be chosen to suppress the band of frequencies in the response which induce instabilities in the time varying system or undesired superharmonic or subharmonic responses. For these bands, the weight on the control effort may be relaxed to minimize the total cost functional as more control effort to suppress the response in these preferential bands is desired without increasing the cost by increased gain over all frequency bands.

The solution of the optimal problem in each frequency band would lead to a Ricatti differential matrix equation as in the classical LQR problem in the respective frequency bands with the frequency band dependent weighting matrices; finally leading to the algebraic Ricatti equation under steady state for each of the frequency bands, based on the assumption of a slowly varying system characteristic matrix $[A(t)]$. Solving for the Ricatti equation for the case of each frequency band produces the required frequency dependent gain matrices to be used to synthesize the control in time domain using Eq. (31).

Real-Time Implementation of Control Algorithm

The real time control algorithm scheme is discussed for a linear time varying system with a state matrix consisting of a time invariant (nominal) and a time varying component.

- Step 1: calculate $[G]_L$ and $[G]_j$ using frequency dependent weighting matrices $[Q]_L$, $[R]_L$ and $[Q]_j$, $[R]_j$; $L, j<u-1$;
- Step 2: set t_0=0, $t_{\text{inc}}=\Delta t$, and define $t_c=t_0+t_{\text{inc}}$;
- Step 3: consider the interval $[t_0, t_c]$;

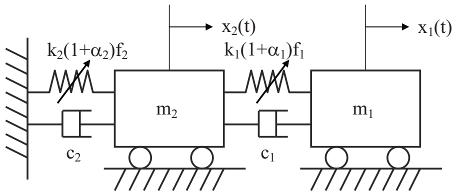

Fig. 1. 2DOF system

- Step 4: record the response $\{x(t): t \in [t_0, t_c]\}$;
- Step 5: decompose $\{x(t)\}$ using MRA based DWT into $\{x(t)\}_L$ and $\{x(t)\}_{d_j}$; $L < j < u-1$;
- Step 6: calculate control input $u(t_c)$ at $t = t_c$ using Eq. (31);
- Step 7: update $t_c = t_c + t_{\text{inc}}$; and
- Step 8: repeat Steps 3–7.

Example Systems

First Example

To illustrate the application of the proposed wavelet based modified LQR control scheme and its effectiveness, a system with known instabilities (given by Den Hartog 1956) has been considered as an example, first of all. The system is a SDOF system with variable stiffness and the free vibration displacement response $x(t)$ (with the overdot denoting differentiation with respect to time) is represented by the mass normalized differential equation

$$\ddot{x} + \omega_n^2 \left[1 + \left(\frac{\Delta k}{k} \right) f(t) \right] x = 0 \tag{35}$$

where ω_n = natural frequency; $(\Delta k / k)$ = maximum ratio of stiffness variability; and $f(t)$ = periodic function of the time representing the stiffness variation. In particular, for the example system considered, the variation of stiffness is assumed to be periodic with frequency ω_k and is given by rectangular steps with binary values. This variation is expressed as

$$\begin{aligned} f(t) &= 1, 2N\pi < \omega_k t < (2N+1)\pi \\ &= -1, (2N+1)\pi < \omega_k t < 2(N+1)\pi \end{aligned} \tag{36}$$

where N being an integer. The instabilities for this particular system for several parametric values have been derived by Den Hartog (1956). Hence, it would be appropriate to see how the controller is able to control the response for this system with instabilities known to exist for certain parameters.

Second Example

To show the effectiveness of the proposed controller in the general context of a multidegree-of-freedom (MDOF) system, a 2DOF time varying system is considered as a second example (shown in Fig. 1). The general equations of motion for the viscously damped forced vibration of the system in a mass normalized form, with absolute displacement response $\{x(t)\} = \{x_1(t) x_2(t)\}^T$, are given by

$$\ddot{x}_1 + 2\varsigma_1 \omega_{n_1} (\dot{x}_1 - \dot{x}_2) + \omega_{n_1}^2 [1 + \alpha_1 f_1(t)](x_1 - x_2) = p_1(t) \tag{37}$$

and

$$\begin{aligned} &\ddot{x}_2 + 2\varsigma_2 \omega_{n_2} \dot{x}_2 - 2\varsigma_1 \omega_{n_1} \mu (\dot{x}_1 - \dot{x}_2) + \omega_{n_2}^2 [1 + \alpha_2 f_2(t)] x_2 \\ &\quad - \mu \omega_{n_1}^2 [1 + \alpha_1 f_1(t)](x_1 - x_2) = p_2(t) \end{aligned} \tag{38}$$

In Eqs. (37) and (38), the damping parameters are $2\varsigma_1 \omega_{n_1} = c_1 / m_1$ and $2\varsigma_2 \omega_{n_2} = c_2 / m_2$; the stiffness parameters are $\omega_{n_1} = \sqrt{k_1 / m_1}$ and $\omega_{n_2} = \sqrt{k_2 / m_2}$; the maximum stiffness variation ratio for the springs attached to the first and the second degree of freedom (i.e., x_1 and x_2, respectively) are $\alpha_1 = \Delta k_1 / k_1$ and $\alpha_2 = \Delta k_2 / k_2$; the functions $f_j(t) (j = 1, 2)$ denote the variation of the stiffness with time for the springs attached to the jth degree of freedom; $p_j(t) (j = 1, 2)$ = excitation accelerations at the jth degree of freedom; and $\mu (= m_1 / m_2)$ = mass ratio.

Results

SDOF System

For numerical simulation of the SDOF system, the parameters for the natural frequency ω_n, the ratio (ω_n / ω_k), and the maximum ratio of stiffness variability $(\Delta k / k)$ are assumed to be 2π rad/s, 1.732, and 0.9, respectively. The response for the system is simulated with an initial displacement of 0.01 m and is seen to diverge in Fig. 2(a). This tallies with the analytical prediction by Den Hartog (1956) as the region with the chosen parameter combination is shown to be unstable (Den Hartog 1956). To investigate the nature of the response and the frequencies induced in the response the Fourier amplitude of the response is plotted in Fig. 2(b). The Fourier spectrum clearly shows that there is considerable amount of energy in the low frequency range (below 4 rad/s) which contributes to the instability in the system. Also, there are additional frequencies (superharmonic) around 10 rad/s, however, relatively low in amplitude. These observations form the basis for choosing the frequency bands for applying the frequency dependent gains to the wavelet domain time-frequency signals of the states.

Implementation of the Control Numerical Algorithm

The orthogonal wavelet basis (db4 with two vanishing moments) proposed by Daubechies is used to decompose the time signals for the different states in the different approximation spaces to represent the signals containing frequencies of desired bands. The Daubechies wavelets are localized in time and frequency to capture the effects of local frequency content in a time signal and allows for fast decomposition and reconstruction using MRA with perfect reconstructing capability. The vibration response signals recorded in real time are decomposed into seven levels with dyadic scales generating seven detail signals corresponding to frequency bands with central frequencies ranging from 3.23 rad/s to 103.52 rad/s and an approximation signal at Level 7. The approximation signal at Level 7 contains frequencies from bands with central frequencies less than 3.23 rad/s. To put emphasis on the low frequency bands, which is a primary reason for inducing the instability in this case, the LQR problem is first solved with relaxed weightage on the control effort, i.e., with $R = 0.1$ and $Q = [I]$. The gains obtained are applied to the filtered time signals for the states in the wavelet domain at the approximation spaces

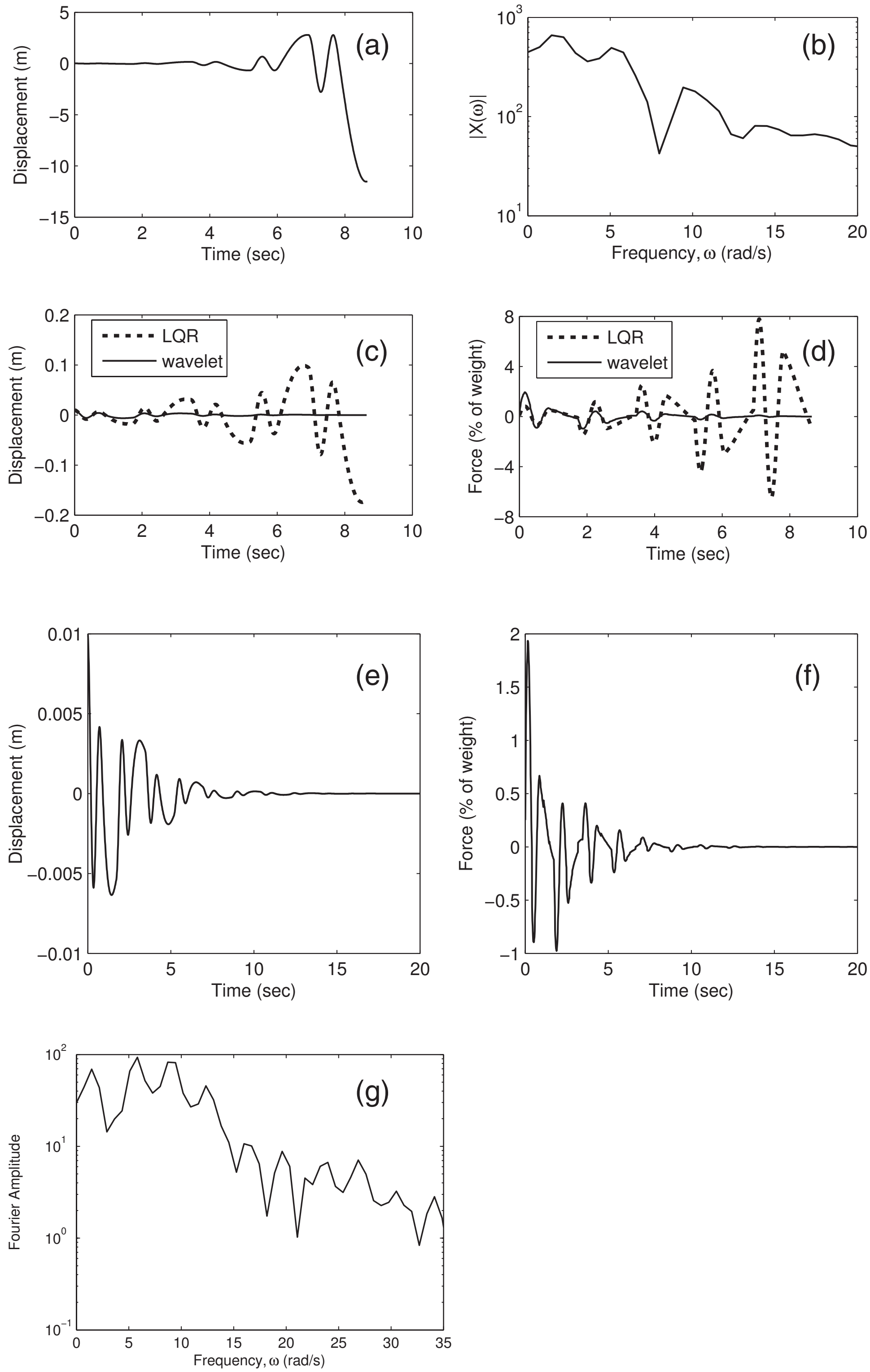

Fig. 2. (a) Uncontrolled response of SDOF system; (b) Fourier amplitude of the uncontrolled response; (c) comparison of controlled response for the LQR and the wavelet controller; (d) control force for the LQR and the wavelet controller; (e) controlled response for wavelet controller (longer duration); (f) control force for wavelet controller (longer duration); and (g) Fourier amplitude of the control force for the wavelet controller

with dyadic frequency bands having central frequencies less than 3.23 rad/s (approximation space for Level 7). The filtered signals in the wavelet domain represent a band of frequencies. Since, by applying the dyadic Daubechies wavelet the subsequent level of approximation corresponds to 6.5 rad/s, it may be approximately stated that the approximation spaces cover 0 rad/s to about 4–5 rad/s. This covers the dominant frequencies introducing instability in the system. Next, to control the vibration associated with the rest of the frequency bands, the LQR problem is again solved with the equal weightings of $R=1$ and $Q=[I]$ for the con-

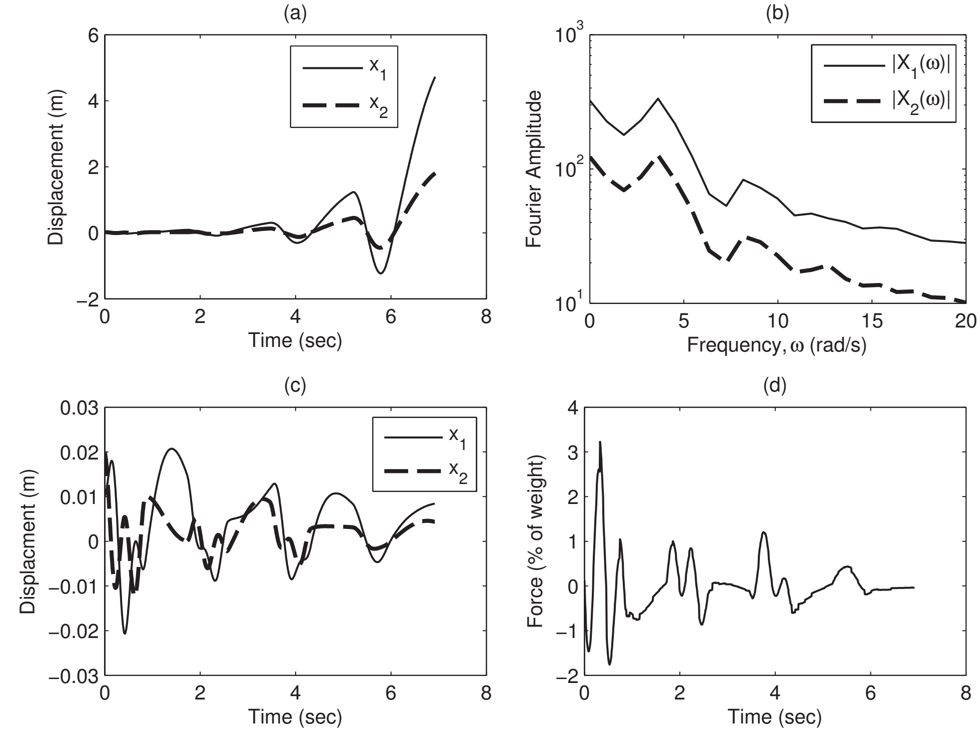

Fig. 3. (a) Free vibration response of 2DOF system; (b) Fourier amplitude of responses of 2DOF system; (c) controlled responses of 2DOF free vibration; and (d) control force for controlling free vibration

trol effort and the response energy, respectively (as is usual in many cases when no special emphasis is desired to be placed on the requirement of additional control effort). The gains obtained are applied to the wavelet based filtered signals of the states for the complement of the approximation space for Level 7. The complement is obtained by subtracting the approximation signal at Level 7 from the recorded signal in the interval concerned. This covers all the frequency bands with central frequencies higher than 3.23 rad/s contained in the vibration signal. The control input is constructed in time domain by a linear combination of the frequency dependent gain weighted filtered signals derived from MRA. This procedure is carried out progressively in time to calculate the control at the current time instant. The signals for the states available up to the current time point is used for wavelet based decomposition of the state signals and synthesis of the control. Since the frequency dependent gains are calculated based on the nominal state matrix $[A_0]$, these are to be calculated offline at the beginning and will subsequently be used for DWT based computation of the control input in real time. As previously mentioned the MRA algorithm for wavelet decomposition and reconstruction is a very fast algorithm with time complexity $\sim O(N)$, where N =length of the data [faster than fast Fourier transform (FFT) for which the time complexity is $\sim O(N \log N)$] and hence, the synthesis of the control is efficient.

The controlled displacement response and the normalized control force (with respect to the weight of the SDOF system) required are plotted in Figs. 2(c and d), respectively. The effectiveness of the control strategy is apparent with the peak normalized control force requirement of less than 2% to control the free vibration response. To compare the performance of the proposed controller with a conventional LQR controller, the control gains are computed with the weightings $R=1$ and $Q=[I]$ and are applied to control the vibration. It is clear that the classical LQR controller fails miserably to control the vibration which is seen to diverge in Fig. 2(c) with a consequence of high building up of the normalized control force requirement. To investigate further if the wavelet controlled response is really stable (as there is a possibility for the response to grow if it close to the stability margin) numerical simulation has been run for a longer duration of time of 60 s and it has been found to be stable. The plots for the displacement response and the control force are shown in Figs. 2(e and f). In addition, the Fourier amplitude of the control force is plotted in Fig. 2(g) showing the concentration of the energy in the low frequency range to control the instability.

2DOF System

Free Vibration

The parameters considered for the simulation of the undamped free vibration of the 2DOF system considered are $\omega_{n_1}=2\pi$ rad/s, $\omega_{n_2}=2.5\pi$ rad/s, $(\omega_{n_1}/\omega_k)=1.732$, and $\mu=0.7$. The maximum ratio of stiffness variability $(\Delta k_1/k_1)$ and $(\Delta k_2/k_2)$ are assumed to be 0.9 each. The responses for the system are simulated by assuming the initial displacements of $x_1=0.01$ m and $x_2=0.02$ m. The plots in Fig. 3(a) clearly show the divergence of the responses and that the system is unstable for the combination of parameters considered for the 2DOF linear time varying system. The plots of the Fourier amplitudes of the responses in Fig. 3(b) confirms the conclusion from Fig. 3(a) and is indicative of the instability with a concentration of the energy content in the range 0–4 rad/s. Hence, the weightings used to obtain the gains corresponding to bands of the frequencies in the approximate range 0–4 rad/s are $R=0.1$ and $Q=[I]$, as was chosen in the case of the SDOF system for preferred frequency bands (approximation Level 7 with dyadic scale). For the complementary space covering all the higher frequencies in the signals, the weighting matrices are kept unaltered as in the case of the SDOF system. The

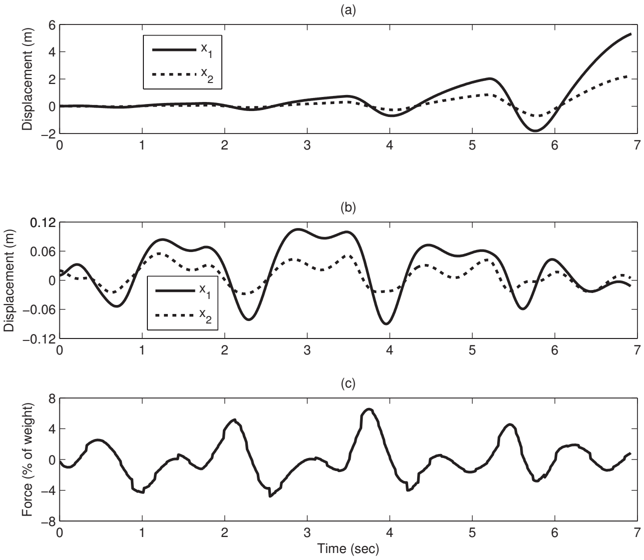

Fig. 4. Harmonic excitation of frequency 7.8 rad/s with viscous damping: (a) uncontrolled response; (b) controlled displacement responses; and (c) control force for harmonic excitation

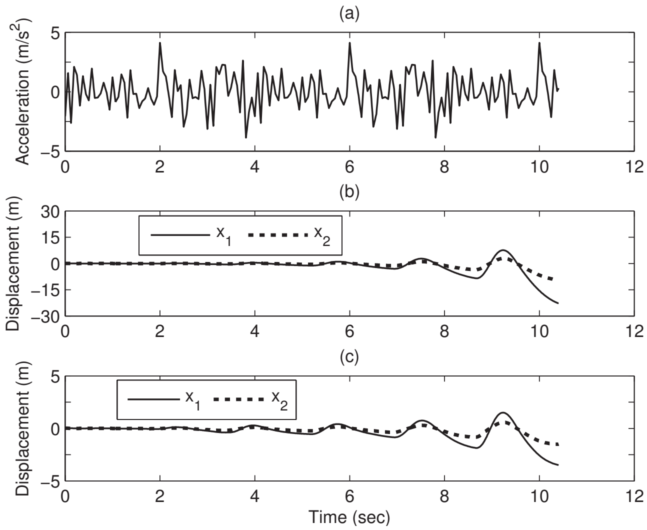

Fig. 5. (a) Random nonstationary excitation simulated with amplitude modulation; (b) responses of 2DOF damped system with $c_1/m_1=c_2/m_2=1$; and (c) responses of 2DOF damped system with $c_1/m_1=c_2/m_2=2$

proposed controller is used to control the response of the 2DOF system with one controller acting on the mass m_1 and the gains are calculated appropriately for the 2DOF system. The controlled displacement responses and the normalized control force (as a percent of the total weight of the 2DOF system) are plotted in Figs. 3(c and d), respectively. The controller is able to control the displacement responses with a peak control force of about 3.2%.

Forced Vibration

On being able to stabilize the free vibration response successfully, the effectiveness of the controller in reducing forced vibration response is examined next. Two cases of forcing are considered; harmonic excitation and simulated nonstationary random excitation.

A harmonic base excitation of frequency 7.8 rad/s (close to ω_{n_2}) is applied to a damped case of the 2DOF system considered previously for the free vibration study. The damping parameters assumed are $c_1/m_1=c_2/m_2=1$. Fig. 4(a) shows that the responses of the viscously damped 2DOF system diverge for the harmonic excitation considered. This clearly happens due to the instability in the linear time-varying system which is unexpected for linear viscously damped systems. The proposed controller is successful in controlling the vibration within the first few seconds of the response [Fig. 4(b)] with normalized control force requirement of less than 6.5% of the total weight [Fig. 4(c)].

The nonstationary random base acceleration is considered next. A band limited excitation is simulated and modulated by a Shinozuka and Sato (1967) type of amplitude modulating function (with parameters $\alpha=4$, $\beta=6.22$ and $\gamma=3.11$ leading to a peak of the modulating function at around 4.99 s) to generate a transient, nonwhite, and nonstationary excitation. A plot of the simulated random excitation is shown in Fig. 5(a) with peak acceleration close to $0.5g$. To investigate the effect of damping and its effectiveness in reducing the unstable response, the forced vibration responses of the 2DOF system with damping coefficients $c_1/m_1=c_2/m_2=1$ and $c_1/m_1=c_2/m_2=2$ are studied and plotted in Figs. 5(b and c), respectively. It is seen from Figs. 5(b and c) that even by doubling the damping, the responses for both the states of the 2DOF system are uncontrollable. Though for a higher viscous damping the build up of the responses are relatively delayed as compared to the case with a lower damping, it is clearly unable to suppress the instability [Fig. 5(c)].

The proposed controller, with similar weights for the case of the previous controllers ($R=0.1$, $[Q]=[I]$ for frequency bands around 0–4 rad/s; and $R=1$, $[Q]=[I]$ for the rest of the frequencies), controls the responses as shown in Fig. 6(a) with peak control force requirement of about 7.5% of the total weight [Fig. 6(b)].

In order to study the effect of multiple controllers in controlling response of the system, two controllers located at the two degrees of freedom of the system are used instead of a single controller applied to mass m_1 as in the previous cases. The weighting matrices in case of two controllers become $R=0.1[I]$, $[Q]=[I]$ for frequency bands around 0–4 rad/s; and $R=[I]$, $[Q]=[I]$ for the rest of the frequency bands. The controlled displacement responses and the control forces as a percentage of the weight of each floor are plotted in Figs. 6(c and d), respectively. The controlled peak displacement particularly for x_2 (at the location where the second controller is placed) reduces significantly by about 50% from about 0.15 m in the case of a single controller to about 0.07 m. Also, the peak control force for each of the controllers is much less than the peak control force requirement for a single controller. The peak control forces for both the controller are about 11.5% of the floor weight which amounts to about 4.7% of the total weight in one controller and 6.7% of the total weight in the other (since mass ratio $m_1/m_2=0.7$). However, it may be noted that the total energy consumed by the two controllers is greater than that of a single controller, as the reduced controlled response and ease in application with the avoidance of problems such as controller saturation comes at a price.

Conclusions

A new type of controller for linear slowly time-varying systems which is based on the multiresolution capability of wavelet analy-

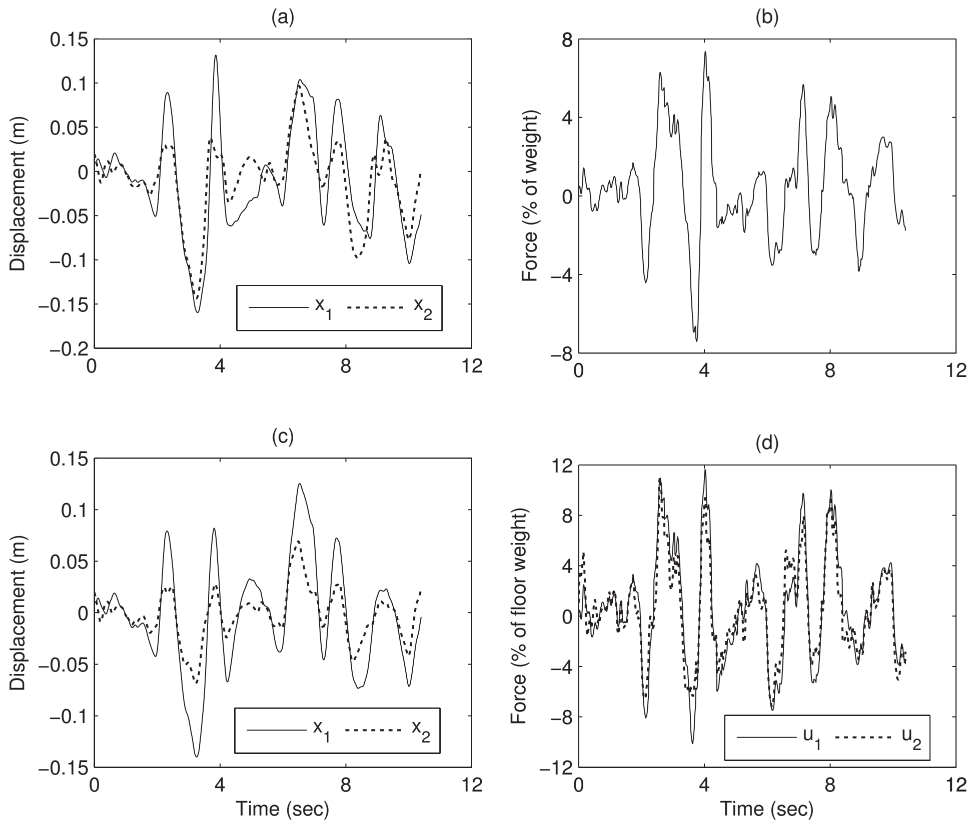

Fig. 6. (a) Controlled response with single controller; (b) control force with single controller; (c) controlled response with two controllers; and (d) control forces with two controllers

sis to produce time-frequency signals is proposed in this paper. The frequency dependent control gain matrices have been used based on the desired emphasis to be placed on different frequency bands for controlling the response of a system. An optimal control problem is formulated and is seen to follow a LQR problem constrained to a band of frequency. The solution of this problem to different frequency bands leads to a suboptimal constrained solution. While the flexibility of choosing frequency dependent weighting matrices provides versatility to the control scheme, the synthesis of the control force in time domain by using linear combination of filtered time-frequency signals weighted by frequency band dependent gains accounts for evolutionary frequency content in a time varying system. The control scheme uses filtered signals in time-frequency domain with the MRA based DWT aiding the process through a fast numerical algorithm and hence is efficient for real-time implementation.

The numerical studies carried out in this paper with the wavelet controller developed using Daubechies wavelet have proved the efficiency of the controller. The proposed controller has been shown to control response of linear time varying systems where the conventional LQR controller has been unsuccessful. The application of the wavelet based multi resolution controller to SDOF and MDOF linear time varying systems for both free and forced vibrations (with harmonic and random excitations) has been successful in efficiently suppressing instabilities even in cases where increased viscous damping has been ineffective. The developed controller in this paper holds promise for extension in suppressing subharmonic and superharmonic responses in nonlinear systems and will be addressed by the writers in forthcoming papers.

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